Math8302 HW 2

Exercises in Chapter 6: 15.5, 15.12, 15.14.
Exercises in Chapter 3: 5.5, 6.1, 7.1, 7.8, 7.13, 6.10
5 problems graded: each problem 15 points.
Completion 25 points.

Ex. 6.15.5. Since the manifold is assumed to orientable, the homology orientation at a point determines the choice in an open set about the point. Suppose $y \in M$ is another point. Choose a path $f : I \to M$ connecting x_0 to y, using the fact that a connected $n-$manifold is path connected. Then the image of the path is compact and so is covered by a finite number of interiors of disks in Euclidean neighborhoods. By taking the inverse image of embedded disks about points of M, we get a covering of I. We can then take a finite subcovering and thus find a sequence $D_1, \cdots, D_k$ of embedded disks so that $D_{i+1} \cap D_i \neq \emptyset$ and $x_0 \in D_1, y \in D_k$. Since the homology orientation at one point in a disk determines the homology orientation at all other points, we inductively see that the homology orientation at x_0 determines it at y.

Ex. 6.15.12. (a) If $x \in H_n^{M,x}$ is in $\tilde{U}$, suppose that W is chosen as above. Then the map $r_{U,y}$ factors as $r_{W,y}r_{U,W}$, where $r_{U,W} : H_n^{M,U} \to H_n^{M,W}$ is induced by inclusion. Since $v_x \in \tilde{U}$, it is of the form $r_{U,x}(\alpha)$. Then if $v_W = r_{U,W}(\alpha)$, we have $r_{W,x}(v_W) = v_x$. Moreover, $r_{W,y}(v_W) = r_{U,y}(\alpha)$ so $r_{W,y}(v_W) \in \tilde{U}$.

(b) First note that given $v_x \in H_n^{M,x}$, we can find a disk neighborhood U of x so that $r_{U,x} : H_n^{M,U} \to H_n^{M,x}$ is an isomorphism for each $y \in U$. Then choose the element $v_U \in H_n^{M,U}$ which maps to v_x under this isomorphism. Then $\tilde{U} = \{r_{U,y}(v_U)\}$ gives a set of this type which contains v_x. ow suppose that $v_x \in \tilde{U}_1 \cap \tilde{U}_2$. Then choose a disk neighborhood $W \subset U_1 \cap U_2$ of x. Then the argument in (a) shows that $\tilde{W} = \{r_{W,y}(v_W) : y \in W\} \subset \tilde{U}_1 \cap \tilde{U}_2$.

(c) We have to establish local triviality. For this we choose for $x \in M$ a small disk neighborhood U with $r_{U,y}$ an isomorphism for each $y \in U$. The the inverse image $p^{-1}(x) = H_n^{M,x} \simeq \mathbf{Z}$. For each $v_x \in p^{-1}(x)$, there is the open set $\tilde{U}$ consisting of the classes $\{v_y = r_{U,y}(v_U)\}$ where $r_{U,y}(v_U) = v_x$. This =is an open set in $\tilde{M}$ which is mapped homeomorphically to U. Since v_U is determined by v_x, and $r_{U,y}$ is an isomorphism for all $y \in U$, these are disjoint open sets for distinct v_x. Hence the set U satisfies the local triviality condition of a covering space.

For p_g, the argument is the same with the distinction that $p_g^{-1}(U)$ is the disjoint union of 2 open sets, one for each generator.

Ex. 6.15.14. (a) If μ_x is a homology orientation, then the map $s : M \to \tilde{M}_g$ given by $s(x) = \mu_x$ is a nonzero section. Conversely, a nonzero section s gives the homology orientation by $\mu_x = s(x)$. We have constructed the basis for the topology on $\tilde{M}_g$ so that this section is continuous–it is just a local homeomorphism using the disk neighborhood of the point.

(b) If we have any other section besides the zero section, then it would map into a component of divisibility d. By using the homeomorphism (which covers the identity map of M, hence an equivalence of covering spaces) between $\tilde{M}_d$ and $\tilde{M}_g$, this gives a section of $\tilde{M}_g$.

(c) Existence of a homology orientation is equivalent to a section of $\tilde{M}_g$, so non-orientability means that there is no such section and so the only section of $\tilde{M}$ is the zero section.

Ex. 3.5.5. (1) f homotopic to g implies $\overline{f}$ is homotopic to $\overline{g}$ and so all the degrees are the same.

(2) The map m_r extends with the same definition to a homeomorphism from D^2 to rD^2, so the extension F of f determines an extension $\overline{F}$ of $\overline{f}$. Hence $deg(f) = deg(\overline{f}) = 0$ by Lemma 3.5.7.

(3) We let $S^1 \times [0,1] \to A(r_1, r_2)$ be determined by $M(z,t) = ((1 - t0r_1 + tr_2)z$. Then $M_0 = m_{r_1}, M_1 = m_{r_2}$. The map uFM is a homotopy between $u(F|_{r_1 S^1})m_{r_1}$ and $u(F|_{r_2 S^2})m_{r_2}$, so $F|_{r_1 S^1})$ and $F|_{r_2 S^2})$ have the same degree.

(4) The composition $ufm_r = f$, so this follows by Lemma 3.5.8.

Ex. 3.6.1. When we take two different radii, then the annular region between the two circles allows us to find a homotopy between the two maps $\underline{v}m_{x,r_1}, \underline{v}m_{x,r_2}$. This is part (3) of Proposition 3.5.10, together with the translation from a neighborhood of 0 to a neighborhood of x. The fact that v only vanishes at x means that v defines a map into $\mathbf{R}^2 \backslash \{0\}$, which is required in defining the homotopy.

Ex. 3.7.1. To check differentiability using S_1, we look at the composition fh_i^{-1}. Using S_2, we use a composition fg_j^{-1}. The relation between these is $fg_j^{-1} = fh_i^{-1}(h_i g_j^{-1})$. Since the map $h_i g_j^{-1}$ is assumed to be a diffeomorphism, one map is differentiable exactly when the other is.

Ex. 3.7.8. (a) The differential structure for P comes from that on S^2, so the covering map is locally just the identity in well chosen local coordinates and otherwise comes from the coordinate transformations for S^2.

(b) In the local coordinates for P coming from S^2, the map $p : S^2 \to P$ becomes the identity, so it is a diffeomorphism.

(c) Since the local coordinates for P come from those of S^2 via the projection, we check differentiability of $f : P \to \mathbf{R}$ by checking the differentiability of $f\overline{h}_i^{-1} = fph_{i1}^{-1}$, which is what we look at to check $fp : S^2 \to \mathbf{R}$ is differentiable.

Ex. 3.7.13. We just extend the vector field radially over the disk we add on each boundary circle so that it is inward pointing, damping it down to zero at the center. For each boundary circle this will add 1 to the index. Thus the original index for such a vector field over M_p is $\chi(M) - p = \chi(M_{(p)})$. Then the index for these vector fields which are outward pointing on the boundary are again given by the Euler characteristic.

Ex. 3.6.10. Let $B(x_i, r_i)$ be a small disk about x_i which excludes the other roots. The the product $w_i = \prod_{i \neq j}(z - x_j)$ will extend over the disk $B(x_i, r_i)$ as a map to $\mathbf{C}\backslash\{0\}$ and $v = (z - x_i)w_i(z)$. By the preceding exercise, the index of v at the singularity x_i is the same as the index of the map $v_i(z) = z - x_i$. But this index is 1 since $\underline{v}_1 m_{x_i, r_i}(z) = z$. The total index is then the sum of the local indices, so is n.